

Note on the Schwarzschild-phantom wormhole

R Lukmanova^{1,*}, A Khaibullina¹, R Izmailov¹, A Yanbekov¹,
R. Karimov¹ and A A Potapov²

¹Zel'dovich International Center for Astrophysics, M. Akmullah Bashkir State
Pedagogical University, Ufa 450000, RB, Russia

²Department of Physics and Astronomy, Bashkir State University, Sterlitamak
Campus, Sterlitamak 453103, RB, Russia

Abstract: Recently, it has been shown by Lobo, Parsaei and Riazi (LPR) that phantom energy with $\omega = p_r/\rho < -1$ could support phantom wormholes. Several classes of such solutions have been derived by them. While the inner spacetime is represented by asymptotically flat phantom wormhole that have repulsive gravity, it is most likely to be unstable to perturbations. Hence, we consider a situation, where a phantom wormhole is somehow trapped inside a Schwarzschild sphere across a thin shell. Applying the method developed by Garcia, Lobo and Visser (GLV), we shall exemplify that the shell can possess zones of stability depending on certain constraints. It turns out that zones corresponding to "force" constraint are more restrictive than those from the "mass" constraint. We shall also enumerate the interior energy content by using the gravitational energy integral proposed by Lynden-Bell, Katz and Bičák. It turns out that, even though the interior mass is positive, the integral implies repulsive energy. This is consistent with the phantom nature of interior matter.

Keywords: Phantom wormhole; thin-shell technique; stability

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1. Introduction

Wormholes are topological tunnels that connect two universes or two distant regions of spacetime. The subject is more of an academic curiosity but nonetheless has received some attention after the influential work by Morris and Thorne [1]. Wormholes have not been observed in any experiment but have not yet been ruled out by observations either. Again, there have been no known experiments specifically designed to observe or rule out such objects. On the other hand, there is enormous work on black holes, especially directed to finding a super-massive black hole at our galactic center¹. There could nonetheless be a modest interest in the topic of wormholes so long as the exterior does not deviate from the Schwarzschild vacuum but allows other matter like phantom in the interior. We then call it phantom wormhole. Since the well known regular Ellis wormhole is unstable [3,4], we look for stable solutions, and hence study the stability of phantom wormholes.

¹We thank an anonymous reviewer for pointing it out.

Phantom wormholes could be interesting in the sense that they are built out of phantom energy defined by equation of state $\omega = p_r/\rho < -1$, which is speculated to be a possible cause driving late-time cosmic acceleration [5]. Both wormholes or phantom equation of state have really not been confirmed by observations. However, mathematical solutions exist. It has been recently shown by Lobo, Parsaei and Riazi (LPR) that wormholes with phantom energy could actually be found as exact solutions (as opposed to artificially engineered ones) of Einstein's equations. Several classes of such solutions have been derived by them. However, since the phantom wormholes are built of the Null Energy Condition (NEC) violating exotic matter (since $p_r + \rho < 0$), they are likely to be unstable though an exact formulation of such instability is still unavailable. We try to imagine a situation, where some kind of stability involving phantom wormholes would still be possible.

To that end, we thought it useful to consider that a phantom wormhole is somehow trapped inside a Schwarzschild sphere across a thin shell. We shall employ a novel and recent formalism to study the stability of such thin shells developed by Garcia, Lobo and Visser (GLV) [6]. We note that the stability of wormholes with phantom energy has been analyzed in [7], but the author consider other configurations of wormholes and took into account only the "mass constraint". At the same time, we know that there is also another constraint, viz., the "force constraint" and is more restrictive in nature, as will be seen later, which allows us to obtain more precise ranges of stability. The formalism was primarily used in Ref.[8] for the stability of the thin-shell Schwarzschild-Ellis wormhole. We shall consider a particular class of asymptotically flat LPR phantom wormholes, viz., that of bounded mass gluing it to a vacuum Schwarzschild exterior. The configuration would resemble a gravastar of certain radius having in its interior repulsive phantom matter and a Schwarzschild vacuum in the exterior. For an outside observer, there would therefore be no distinction between a true Schwarzschild mass and a gravastar of this kind.

In the present paper, we shall extend the stability analysis of the Schwarzschild - phantom thin-shell wormhole to include also the "external force" term [6], and delineate the zones of stability. In addition, we shall calculate the interior energy content by using the gravitational energy integral proposed by Lynden-Bell, Katz and Bičák [10,11]. It turns out that, even though the interior mass is positive, the integral implies repulsive energy in the interior. This is consistent with the fact that the interior mass is phantom in nature.

The paper is organized as follows: In Sec.2, we shall outline the GLV method. In Sec.3, we outline the features of the LPR phantom wormhole under consideration and in Sec.4, we enumerate the interior energy constant. In Sec.5, we build the composite object gluing together at some radius the phantom wormhole and the exterior Schwarzschild spacetime. Results are discussed in Sec.6. We take units such that $G = c = 1$, unless specifically restored.

2. The GLV method

We shall only cite their results that will be used here and take the liberty

to restate the new concepts they have developed. For details of their ingenious arguments and calculations, the reader is asked to read the original paper [6]. They take the spacetimes on two \pm sides of the thin shell as

$$ds^2 = -e^{2\Phi_{\pm}(r_{\pm})} \left[1 - \frac{b_{\pm}(r_{\pm})}{r_{\pm}} \right] dt_{\pm}^2 + \left[1 - \frac{b_{\pm}(r_{\pm})}{r_{\pm}} \right]^{-1} dr_{\pm}^2 + r_{\pm}^2 d\Omega_{\pm}^2. \quad (1)$$

The method allows any two arbitrary spherically symmetric spacetimes to be glued together by cut and paste procedure. Thus, for the static and spherically symmetric spacetime, the single manifold \mathcal{M} is obtained by gluing two bulk spherically symmetric spacetimes \mathcal{M}_+ and \mathcal{M}_- at a timelike junction surface Σ , i.e., at $f(r, \tau) = r - a(\tau) = 0$. The surface stress-energy tensor may be written in terms of the surface energy density σ and the surface pressure P as $S_{ij} = \text{diag}(-\sigma, P, P)$. GLV work out the general conservation law

$$\frac{d(\sigma A)}{d\tau} + P \frac{dA}{d\tau} = \Xi A \dot{a}, \quad (2)$$

where $\dot{a} = \frac{da}{d\tau}$, the shell surface area $A = 4\pi a^2$ and there is an entirely new term

$$\Xi = \frac{1}{4\pi a} \left[\Phi'_+(a) \sqrt{1 - \frac{b_+(a)}{a}} + \dot{a}^2 + \Phi'_-(a) \sqrt{1 - \frac{b_-(a)}{a}} + \dot{a}^2 \right]. \quad (3)$$

The first term in Eq.(2) represents the variation of the internal energy of the shell, the second term is the work done by the internal force of the shell. The right hand side is the net discontinuity in the conservation law of the surface stresses of the bulk momentum flux and is physically interpreted as the work done by external forces on the thin shell. In short it is the "external force" term occurring due to $\Phi'_{\pm} \neq 0$. This is a new concept not noted so far in the literature. Only when $\Phi'_{\pm}(a) = 0$, we have $\Xi = 0$, and then one recovers the familiar conservation law on the shell.

Assuming integrability of σ [6], which allows $\sigma = \sigma(a)$, it is possible to define the mass of the thin shell of exotic matter residing on wormhole throat as [6]

$$m_s(a) = 4\pi\sigma(a)a^2. \quad (4)$$

GLV derived the workable master inequalities about stability around a given radius a_0 after long calculations, which are the constraint from the "mass"

$$m_s''(a_0) \geq \frac{1}{4a_0^3} \left[\frac{[b_+(a_0) - a_0 b'_+(a_0)]^2}{[1 - b_+(a_0)/a_0]^{3/2}} + \frac{[b_-(a_0) - a_0 b'_-(a_0)]^2}{[1 - b_-(a_0)/a_0]^{3/2}} \right] + \frac{1}{2} \left[\frac{b''_+(a_0)}{\sqrt{1 - b_+(a_0)/a_0}} + \frac{b''_-(a_0)}{\sqrt{1 - b_-(a_0)/a_0}} \right] \quad (5)$$

and when $\Phi'_{\pm}(a_0) \leq 0$, the constraint from the "external force"

$$\begin{aligned}
[4\pi\Xi(a)a]''|_{a_0} \geq & \left[\Phi_+'''(a)\sqrt{1-b_+(a)/a} + \Phi_-'''(a)\sqrt{1-b_-(a)/a} \right]|_{a_0} \\
& - \left[\Phi_+''(a)\frac{(b_+(a)/a)'}{\sqrt{1-b_+(a)/a}} + \Phi_-''(a)\frac{(b_-(a)/a)'}{\sqrt{1-b_-(a)/a}} \right]|_{a_0} \\
& - \frac{1}{4} \left[\Phi_+'(a)\frac{[(b_+(a)/a)']^2}{[1-b_+(a)/a]^{3/2}} + \Phi_-'(a)\frac{[(b_-(a)/a)']^2}{[1-b_-(a)/a]^{3/2}} \right]|_{a_0} \\
& - \frac{1}{2} \left[\Phi_+'(a)\frac{(b_+(a)/a)''}{\sqrt{1-b_+(a)/a}} + \Phi_-'(a)\frac{(b_-(a)/a)''}{\sqrt{1-b_-(a)/a}} \right]|_{a_0}. \quad (6)
\end{aligned}$$

Similar, but not the same, force constraint appears for $\Phi'_\pm(a_0) \geq 0$ as well. We shall not quote it here as our example has $\Phi'_\pm(a_0) \leq 0$. Such a force constraint however disappears if $\Phi'_\pm(a_0) = 0$.

Eqs.(5) and (6) are the ones that will be used in the case of the LPR wormhole.

3. The LPR phantom wormhole

We shall consider here only the solution with bounded mass. To be consistent with notation with the above section, we shall change the constant in the LPR wormhole [9] such that it reads

$$ds^2 = - \left[1 + \frac{\lambda r_0}{r} \right]^{1-\frac{1}{\lambda}} dt^2 + \frac{dr^2}{1 - \frac{r_0}{r} \left(\frac{\lambda r_0}{r} + 1 - \lambda \right)} + r^2 d\Omega^2, \quad (7)$$

where λ is a constant and r_0 is the minimum radius. According to equation of state $\lambda\omega = -1$, we get the following range $-1 < \lambda < 0$. The mass function is

$$m(r) = \frac{\lambda r_0}{2} \left(\frac{r_0}{r} - 1 \right). \quad (8)$$

The throat appears at

$$r_{\text{th}} = r_0 \text{ and } \lambda r_0, \quad (9)$$

but we ignore the second radius because it is negative. Thus the wormhole spacetime is defined for $r_0 < r < \infty$. The shape function and the redshift function respectively are

$$b_- = r_0 \left(\frac{\lambda r_0}{r} + 1 - \lambda \right), \quad \Phi_- = \frac{1}{2} \ln \left[\frac{\left[1 + \frac{\lambda r_0}{r} \right]^{1-\frac{1}{\lambda}}}{1 - \frac{r_0}{r} \left(\frac{\lambda r_0}{r} + 1 - \lambda \right)} \right]. \quad (10)$$

Using the Einstein field equation, $G_{\mu\nu} = 8\pi T_{\mu\nu}$, the (orthonormal) stress-energy tensor components in the bulk are given by

$$\rho = -\frac{\lambda r_0^2}{8\pi r^4}, \quad (11)$$

$$p_r = -\frac{r_0^2}{8\pi r^4}, \quad (12)$$

$$p_t = \frac{r_0^2(4r + (\lambda^2 + 4\lambda - 1)r_0)}{32\pi r^4(r + \lambda r_0)}. \quad (13)$$

Here ρ is the energy density, p_r is the radial pressure, and p_t is the lateral pressure measured in the orthogonal direction to the radial direction. We thus obtain the NEC violating condition

$$\rho + p_r = -\frac{(1 + \lambda)r_0^2}{8\pi r^4} < 0, \quad \forall r. \quad (14)$$

4. Energetics in the phantom wormhole

We shall use this form of the metric for gluing with the Schwarzschild black hole exterior and place the mass in the interior of the thin shell. Using (7), one has

$$\rho = \frac{1}{8\pi r^2} \frac{db}{dr} = -\frac{\lambda r_0^2}{8\pi r^4} > 0, \quad \Omega_{\text{WEC}} = \frac{1}{2} \int_{r_0}^{\infty} \rho r^2 dr = -\frac{\lambda r_0}{16\pi} > 0, \quad (15)$$

where Ω_{WEC} is the total amount of Weak Energy Condition (WEC) violating matter, which, in this case, is positive. To understand the nature of gravity (whether attractive or repulsive) due to the interior mass defined by Eq.(8), let us consider the Lynden-Bell, Katz and Bičák [10,11] form of gravitational energy E_G defined by

$$E_G = \Omega_{\text{WEC}} - E_M, \quad (16)$$

where E_M is the sum of other forms of energy like rest energy, kinetic energy, internal energy etc. and is defined by

$$E_M = \frac{1}{2} \int_{r_0}^{\infty} \rho (g_{rr})^{1/2} r^2 dr. \quad (17)$$

In the present case, we find that

$$\begin{aligned} E_M &= \frac{\sqrt{\lambda} r_0 \left(\pi - 2 \arcsin \left[\frac{1}{\sqrt{1+\lambda}} \right] \right)}{16\pi} < 0 \\ \Rightarrow E_G &= -\frac{\sqrt{\lambda} r_0 \left(\sqrt{\lambda} + \pi - 2 \arcsin \left[\frac{1}{\sqrt{1+\lambda}} \right] \right)}{16\pi} > 0, \end{aligned} \quad (18)$$

which implies that, even though the mass in Eq.(8) is positive, the gravitational energy E_G of the interior matter is positive, hence *repulsive* in nature (Fig.1).

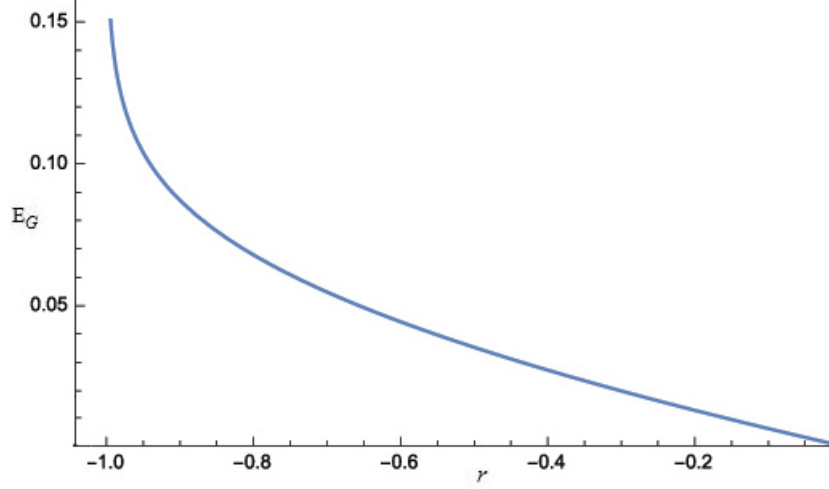


Figure 1: Energetics of the LPR wormhole, where $\lambda \in [-1, 0]$

This only points to the repulsive nature of phantom energy. This interior repulsion is balanced by the exterior attraction due to the Schwarzschild mass at a place where the phantom wormhole's surface is formed. Similar constructions exist in the literature [12-17]. For example, the static spherically symmetric vacuum condensate star (gravastar) devised by Mazur and Mottola [16] has an isotropic de-Sitter vacuum in the interior, the matter marginally satisfying the NEC and strictly violating the Strong Energy Condition (SEC). It has a Schwarzschild exterior ($p = 0, \rho = 0$) of mass M . The difference with our case is that we have a phantom wormhole interior instead of the de-Sitter vacuum.

5. Schwarzschild - phantom wormhole

We glue the horizonless LPR phantom wormhole with Schwarzschild exterior at some given radius $r = a_0 > 2M$, r_0 (throat radius of phantom wormhole), that is, we just join the two spacetimes, LPR and Schwarzschild, at a radius above the Schwarzschild horizon, i.e., at $r > r_{\text{hor}} = 2M$. The interior regions $r \leq 2M$, r_0 are surgically excised out from respective spacetimes because we don't want the presence of any horizon in the resultant wormhole. This a_0 is the radius about which linear spherical perturbations are assumed to take place. Casting the Schwarzschild metric in the GLV form (1), we get

$$b_+ = 2M, \quad \Phi_+ = 0 \quad (19)$$

and similarly casting phantom wormhole metric (7) in the form of GLV metric (1), we get Eqs.(10).

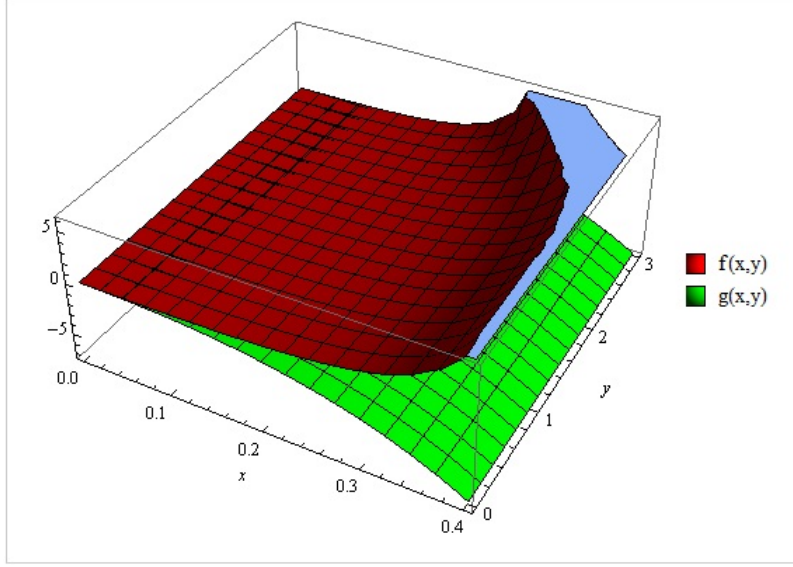


Figure 2: Stability zone for the Schwarzschild - Phantom wormhole with a bounded mass function thin shell. The ranges are defined by $x = \frac{M}{a_0}$, $y = \frac{r_0}{M}$, where $x \in [0, 0.4]$, $y \in [0, 3]$ and $\lambda = -0.5$.

Since $-1 < \lambda < 0$, we have $\Phi'_\pm \leq 0$ for $a_0 > r_0$, and there will occur the effect of "external force" influencing the thin shell motion. Putting the above functions in the inequalities (5,6), and defining

$$x = \frac{M}{a_0}, \quad y = \frac{r_0}{M}, \quad (20)$$

we find, respectively

$$4a_0 m_s''(a_0) \geq f(x, y) = \frac{4}{(1-2x)^{3/2}} + \frac{4\lambda y^2}{\sqrt{(1-xy)(1+\lambda yx)}} + \frac{y^2(1-\lambda+2\lambda yx)^2}{((1-xy)(1+\lambda yx))^{3/2}}, \quad (21)$$

$$8a_0^3 [4\pi\Xi(a)a]''|_{a_0} \geq g(x, y) = \frac{-x^2(1+\lambda)}{((1-x)(1+\lambda x))^{5/2}} [(48 + 12\lambda^2 x^4 + 4x(\lambda - 1)(19 - 10x^2\lambda + \sqrt{(1-x)(1+\lambda x)})) + x^2(31 + 31\lambda^2 - 2\lambda(55 + 6\sqrt{(1-x)(1+\lambda x)}))] \quad (22)$$

6. Results and discussion

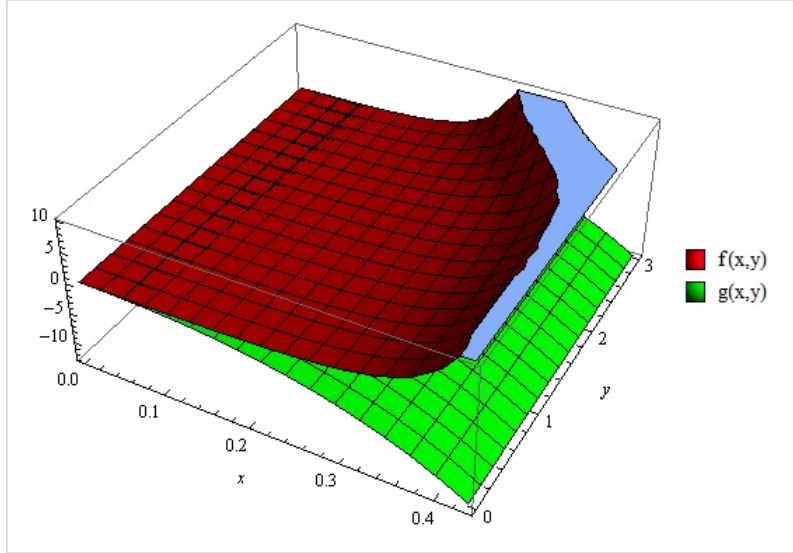


Figure 3: Stability zone for the Schwarzschild - Phantom wormhole with a bounded mass function thin shell. The ranges are defined by $x = \frac{M}{a_0}$, $y = \frac{r_0}{M}$, where $x \in [0, 0.4]$, $y \in [0, 3]$ and $\lambda = -0.2$.

A possible candidate for the accelerated expansion of the Universe is speculated to be phantom energy, a cosmic fluid governed by an equation of state of the form $\omega = p_r/\rho < -1$, which consequently violates the null energy condition. We have analyzed the physical properties and characteristics of thin-shell phantom wormholes gluing with exterior Schwarzschild spacetime. The reason for choosing Schwarzschild exterior is two fold: first, this exterior spacetime is the simplest and most well discussed, especially in the context of the existence of a black hole in the galactic center. Second, the speculation of phantom energy curling up by some high energy process into a Schwarzschild-like star is by itself of interest.

We have analyzed the stability regions of such configurations by including the effects of "external forces", in addition to that of the "mass" term. The "force constraint" is just a momentum flux across the shell and is included here only for completeness. We enumerated the interior energy content by using the gravitational energy integral by Lynden-Bell, Katz and Bičák [10,11]. It turns out that, even though the interior mass is positive, the integral is positive implying repulsive energy in the interior. This is consistent with the fact that the mass is phantom in nature. To make stability analyses physically meaningful, one should be able to delineate the possible parameter ranges for which stability can be achieved. The general and unified GLV stability analysis provides an excellent way to achieve this goal. Our example above is interesting in the sense that the outer mouth has a Schwarzschild mass ($M \neq 0$) as viewed by observers

in the first asymptotic region and the inner mouth has nonzero asymptotic mass as viewed by observers in the second asymptotic region. So the configuration resembles that of a gravastar.

The stability condition can be expected as an explicit inequality involving the second derivative of the "mass" of the throat, $m_s''(a)$. In the absence of "external forces", this inequality is given by (5), and this is the only stability constraint one requires. However, once one has external forces ($\Phi'_\pm(a_0) = 0$), there arises additional constraints in the form of inequality (6), which is imposed from the external force over those from the linearized perturbations around static solutions of the Einstein field equations, as worked out above.

Our results show the following: Fig.2 $f(x, y)$ refers to the effect on stability of the motion of thin-shell due to its "mass" and $g(x, y)$ – effect due to the "external force" ($\lambda = -0.5$). The region above the surface which is given by function $g(x, y)$ is the stable region. It can be seen that, as x increases, the stability region gradually diminishes to zero near $x \sim \frac{1}{3}$. Thus the thin-shell motion becomes unstable very near the Schwarzschild horizon but this phenomenon is insensitive to values of y at any fixed value of x . The region above the surface which is given by function $f(x, y)$ is the stable region. We see that, when x increases from 0 to $\frac{1}{3}$, the stability region steadily diminishes finally to zero near higher values of $x \sim \frac{1}{3}$. However, as y increases from 0 to 3, the region of stability steadily decreases until it diminishes completely near the higher end, $y \sim 3$.

Thus the sensitivity in the y -direction, absent for $g(x, y)$, is revealed for $f(x, y)$. Note that the "mass" constraint (5) does not involve Φ_\pm and its derivatives, while the "force" constraint (6) involves them and in fact exists due to them. Similar results are obtained with $\lambda = -0.2$ and shown in Fig.3. Our analysis confirms that the force constraint provide a more complete stability zone than those from the mass constraint. This informative picture has been available thanks only to the effect of the newly developed "external force" constraint by GLV.

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